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1997 J. Phys. A: Math. Gen. 30 L155

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## LETTER TO THE EDITOR

**Deterministic spin models with a glassy phase transition**

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Received 18 October 1996

**Abstract.** We consider the infinite-range deterministic spin models with Hamiltonian  $H = \sum_{i,j=1}^N J_{i,j} \sigma_i \sigma_j$ , where  $\mathbf{J}$  is the quantization of a chaotic map of the torus. The mean-field Thouless–Anderson–Palmer (TAP) equations are derived by summing the high-temperature expansion. They predict a glassy phase transition at the critical temperature  $T \sim 0.8$ .

A class of infinite-range deterministic Ising spin models with glassy behaviour in numerical simulation has recently been identified [1–3]: unlike the random coupling case, however, the mean-field equations [4] (hereafter the Parisi–Potter (PP) equations) are different from the standard Thouless–Anderson–Palmer (TAP) equations [5] and their linearization does not determine a critical temperature for the glassy transition.

The most interesting representatives of the class are the sine (equivalently, cosine) models, with Hamiltonian

$$H = -\frac{1}{2} \sum_{i,j=1}^N J_{i,j} \sigma_i \sigma_j \quad (1)$$

where the coupling matrix  $\mathbf{J}$  is (twice the uppermost left block) of the discrete sine (cosine) Fourier transform

$$J_{i,j} = \frac{2}{\sqrt{2N+1}} \sin\left(\frac{2\pi ij}{2N+1}\right) \quad i, j = 1, \dots, N. \quad (2)$$

In fact, the ground state of  $H$  can be explicitly computed [3] if  $2N+1$  is prime with  $N$  odd: this leads to the detection of a first-order crystalline phase transition (at a higher critical temperature than the glassy one [2,3]), whose relevance on the glassy behaviour of the system is discussed in [4, section 3]. In turn, the matrix  $\mathbf{J}$  coincides [6] with (the imaginary part of) the unitary propagator quantizing the discrete dynamics generated by the unit symplectic matrix  $\mathbf{S} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  acting as a Hamiltonian map over the 2-torus  $\mathbb{T}^2$  (the operator quantizing a Hamiltonian map of the torus is a  $N \times N$  unitary matrix [7, 8],  $N$  being the inverse of the Planck constant<sup>†</sup>: in this context the thermodynamic limit  $N \rightarrow \infty$  is formally equivalent to the classical limit). This algebraic identity suggests as more natural candidates for detecting deterministic glassy behaviour the coupling matrices defined by the quantization of hyperbolic maps over  $\mathbb{T}^2$ . The corresponding discrete dynamical systems

<sup>†</sup> The physical intuition is: the phase space has volume 1, and can accommodate at most  $N$  quantum states of volume  $\hbar$ , so that  $N\hbar = 1$ .

are indeed chaotic, while  $\mathbf{S}$  generates a period-4 system. Here we consider the matrices  $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  with  $a = d = 2g$ ,  $b = 1$ ,  $c = 4g^2 - 1$ ,  $g \in \mathbb{N}$ , which admit two positive eigenvalues  $\lambda_1 > 1$ ,  $\lambda_2 < 1$ . The quantization of  $\mathbf{A}$  is [7–9] the unitary  $N \times N$  matrix

$$V(A)_{jk} = C_N N^{-1/2} \exp[(2\pi i/N)(gj^2 - jk + gk^2)] \quad (3)$$

with  $|C_N| = 1$ . The present models are thus defined by the Hamiltonians  $H_A(\sigma) = \sum_{j,k} J(A)_{jk} \sigma_j \sigma_k$  with  $\mathbf{J}(\mathbf{A}) = \text{Re}[\mathbf{V}(\mathbf{A})]$ , i.e.

$$J(A)_{jk} = C_N N^{-1/2} \cos[(2\pi i/N)(gj^2 - jk + gk^2)]. \quad (4)$$

Now  $\bar{\mathbf{V}}(\mathbf{A}) = \bar{\mathbf{V}}^T(\mathbf{A}) = \mathbf{V}^{-1}(\mathbf{A})$ , whence  $\dagger \sigma(\text{Re}[\mathbf{V}(\mathbf{A})]) = \text{Re}[\sigma(\mathbf{V}(\mathbf{A}))]$ . Hence the equidistribution [10] of the spectrum  $\sigma(\mathbf{V}(\mathbf{A}))$  over the unit circle for  $N \rightarrow \infty$  with an eigenvalue at 1 for all  $N$  implies the same properties for  $\sigma(\mathbf{J}(\mathbf{A}))$  in  $[-1, 1]$ .

Our claim is that these deterministic models behave, at least in the mean-field approximation, more closely to the random models, such as the Sherrington–Kirkpatrick (SK) model [11]. We will indeed compute in closed form (at the thermodynamic limit  $N \rightarrow \infty$ ) the Gibbs (i.e. magnetization dependent) free energy  $\beta\Phi$ . The result is

$$\beta\Phi = 1/2 \sum_{i=1}^N \{(1 + m_i) \ln(1 + m_i)/2 + (1 - m_i) \ln(1 - m_i)/2\} \\ - \beta/2 \sum_{i,j=1}^N J(A)_{ij} m_i m_j - NG(\beta(1 - q)) \quad G(\beta) = \beta^2/(8 + 4\beta^2) \quad (5)$$

where  $m_l: l = 1, \dots, N$  is the magnetization at site  $l$  and  $q = 1/N \sum_{l=1}^N m_l^2$ , the Edwards–Anderson order parameter (as in [4],  $G(\beta)$  does not depend on the particular choice of  $\mathbf{J}(\mathbf{A})$ ). The stationarity condition of the Gibbs free energy [12] yields the mean-field equations of the model (believed exact because of its infinite range)

$$Q_i \equiv \tanh^{-1} m_i + 2\beta G'(\beta(1 - q)) m_i - \beta \sum_j J(A)_{ij} m_j = 0. \quad (6)$$

Unlike the PP equations (but like the TAP [5]) these equations can be solved by linearization near  $q = 0$ : any eigenvector for the eigenvalue 1 of  $\mathbf{J}(\mathbf{A})$  yields a solution if  $2\beta^2/(2 + \beta^2)^2 - \beta + 1 = 0$ . Its only positive zero  $\beta \sim 1.25$  determines the critical temperature  $T_c \sim 0.8$ . For  $T > T_c$  (6) has only a trivial solution, corresponding to the paramagnetic phase. The phase transition is glassy because, as we will see,  $1/N \sum_{l=1}^N m_l \rightarrow 0$ ,  $N \rightarrow \infty$  at the critical point, while for  $T < T_c$  the analysis of [4] can be applied to yield the existence of an exponentially growing number of solutions of the TAP equations (6). This fact and (5) implies that all such solutions have the same free energy. The thermodynamics is thus independent of their number, the multiplicity of the eigenvalue 1, which depends in a sensitive way on the integer  $N$  [10].

To obtain the mean-field equations (6) we resum the high-temperature expansion by the same procedure as in [4]. The function  $G(\beta)$  has a simpler form in this case because there is only one class of non-vanishing diagrams at the thermodynamic limit.

Consider the Helmholtz free energy  $F(\beta)$ : if  $Z(\beta) = \sum_{\{\sigma\}} \exp \beta H_A(\sigma)/2 = \exp -\beta F(\beta)$  is the partition function (at site-dependent magnetic field  $h_i = 0$ ) we have [13]

† This holds only if  $a, b, c, d$  are as above, and motivates our choice of hyperbolic matrices.

$$\begin{aligned}
 e^{-\beta F} &= (2/\pi)^{N/2} / \det^{\frac{1}{2}}(\beta J) \int_{\mathbb{R}^N} \exp \left\{ \langle (\beta J)^{-1} \phi, \phi \rangle + \sum_i \log \cosh(\phi_i + h_i) \right\} d^N \phi |_{h_i=0} \\
 &= 2^N \pi^{-N/2} \int_{\mathbb{R}^N} \exp \left\{ -\langle x, x \rangle + \sum_{i=1}^N \log \cosh[\sqrt{2\beta} \sum_h ((J^{1/2})_{ih} x_h)] \right\} d^N x. \quad (7)
 \end{aligned}$$

The high-temperature expansion for  $\beta F(\beta)$  is generated out of the integration of the expansion of the  $\exp \log \cosh x$ . The well known diagrammatic representation of the  $n$ th-order term is obtained [13] by drawing all diagrams with  $n$  links,  $2 \leq j+1 \leq n+1$  vertices and no external legs, whose individual contribution is:

- (1) for any link between two consecutive vertices  $l \neq k$  a factor  $\beta J_{lk}$ ;
- (2) for any vertex with  $m$  links the cumulant  $u_m$ , i.e. the  $m$ th coefficient of the Taylor expansion of  $\log \cosh x$  ([13, p 414]);
- (3) any diagram has to be divided by its order of symmetry.

The contribution of each individual diagram  $D$  at order  $\beta^n$  is indeed

$$|D| = U(D) S(D)^{-1} \sum_{r_1, \dots, r_{j+1}} J_{r_1, r_2}^{\alpha_1} \cdot J_{r_2, r_3}^{\alpha_2} \cdots J_{r_j, r_{j+1}}^{\alpha_j} \quad j = 1, \dots, n. \quad (8)$$

Here  $j+1$  is the number of vertices,  $\alpha_1 + \dots + \alpha_j = n$ ;  $n-j+1 = n-1, \dots, 1$  is the number of loops,  $\alpha_i \geq 1$  the number of links between consecutive vertices,  $S(D)$  the symmetry factor and  $U(D) = u_1(\alpha_1) u_2(\alpha_1 + \alpha_2) \dots u_j(\alpha_{j-1} + \alpha_j) u_{j+1}(\alpha_j)$ . Note that  $r_{n+1} = r_1$  for  $j = n$ . We now verify that, at the thermodynamic limit  $N \rightarrow \infty$ :

- (a) all diagrams for  $n = 2p + 1$  vanish;
- (b) for  $n = 2p$  the only surviving diagram  $D_p$  is the one with  $p+1$  vertices,  $p$  loops and two links between consecutive loops:



To prove (a) and (b), first recall the basic estimate fulfilled by the Gauss sums [14]

$$\left| \sum_{s_1, \dots, s_l=1}^N \exp(2\pi i/N) g(s_1, \dots, s_l) \right| \leq CN^{l/2} \quad (9)$$

where  $g$  is any quadratic form in the  $l$  integers  $s_1, \dots, s_l$  with integer coefficients and  $C$  a constant independent of  $g$  and  $N$ . By (3) and (4), the sum in (8) amounts for any fixed  $1 \leq j \leq n$  to  $2^n$  Gaussian sums over  $r_1, \dots, r_j$  divided by  $N^{n/2}$ . We now argue that all these Gaussian sums are, once divided by  $N^{n/2}$ , of order 1 or less except one: that with  $n = 2p$ ,  $\alpha_1 = \dots = \alpha_p = 2$ , and summand independent of  $r_1, \dots, r_{p+1}$ , generated by the constant term in the expansion of  $\prod_k J_{r_k, r_{k+1}}^2$ . This sum is clearly equal to  $N^{p+1} N^{-p} 2^{-p} = N 2^{-p}$ . Any other (divided) sum is down by at least  $N^{-1}$ : each summation over  $q$  indices is estimated by  $N^{q/2}$  if the summand is a Gaussian and by  $N^q$  if the summand is constant, which case shows up whenever  $\alpha_i$  is even for at least one  $i$ ; on the other hand, any power of  $\alpha_i$  in excess of one increases by one the number of loops and thus reduces by one the number of vertices and hence of summation indices. Thus for  $n = 2p + 1$  the most divergent sums behave as  $N^{p+1/2}$ : this happens either for  $\alpha_i = 1$  for all  $i$  so that  $j = 2p$ , or for  $\alpha_1 = \dots = \alpha_{p-1} = 2, \alpha_p = 3$  (and permutations thereof) so that  $j = p + 1$ . The prefactor  $N^{-(p+1/2)}$  yields the estimate  $O(1)$ . For  $n = 2p$  the second most divergent sums have the same behaviour: again they take place for  $\alpha_i = 1$  for all  $i$  and  $j = 2p - 1$  or for  $\alpha_1 = \dots = \alpha_p = 2$  with the summand depending but on two indices (up to permutations). Now the only surviving  $D_p$  has two vertices with two links (the extrema)

and  $p - 1$  vertices with four links (all the remaining ones). Since  $u_2 = 1, u_4 = -2$ ,  $U(D_p) = (-2)^{p-1}$ . Moreover, the symmetry factor  $S(D_p)$  is  $2 \cdot 2^p$  (to account for the interchange of the external vertices and of any pair of links between the internal vertices). Hence  $U(D_p)/S(D_p) = (-1)^{p-1}/4$  and by (8) we have  $|D_p| = (-1)^{p-1} \cdot 2^{-p-2}$ . Summing up we obtain the Helmholtz free energy:

$$-\beta F(\beta) - N \ln 2 = N \sum_{p=1}^{\infty} (-1)^{p-1} \cdot 2^{-p-2} \beta^{2p} = N \beta^2 / (8 + 4\beta^2) = NG(\beta). \tag{10}$$

As in [4], expanding  $-\beta F(\beta)$  to first order we recover the SK Helmholtz free energy  $N \ln 2 + \beta^2/4 \sum_{i,j} J_{i,j}^2$  because here  $\sum_{i,j} J_{i,j}^2 = N/2 + O(1)$  for  $N$  large.

To obtain the Gibbs free energy we have to perform the Legendre transform

$$\Phi(\beta, m_i) \equiv \max_{h_i} \left[ F(\beta, h_i) - \sum_i h_i m_i \right] \tag{11}$$

out of the ( $h_i$  dependent) expansion of (7) in powers of  $\beta$ . To this end we simply take over the PP argument because we are summing over a subclass of the ‘cactus’ diagrams considered in [4] within the same assumption of self-averaging, namely  $m_k^2 = N^{-1} \sum_{l=1}^N m_l^2 + o(1)$  as  $N \rightarrow \infty$ . This yields (5), and hence the mean-field equations (6) and the critical temperature.

It remains to prove that  $N^{-1} \sum_{l=0}^{N-1} m_l^i \rightarrow 0$  as  $N \rightarrow \infty$  if  $\mathbf{J}(\mathbf{A})m_k = m_k$ . Any such eigenvector defines indeed a critical (staggered) magnetization distribution; furthermore the vanishing of the total magnetization represents the necessary condition both for the glassy nature of the transition as well as for self-averaging. Setting  $\phi_0 = N^{-1/2}(1, 1, \dots, 1)$  we have  $N^{-1} \sum_{l=0}^{N-1} m_l^i = \langle m_k, \phi_0 \rangle = \langle m_k, \mathbf{V}(\mathbf{A})^l \phi_0 \rangle \forall l \in \mathbb{N}$ . On the other hand, [10]  $\mathbf{V}(\mathbf{A})$  tends (weakly) to a unitary operator on  $L^2(-\pi, \pi)$  with Lebesgue spectrum on the unit circle as  $N \rightarrow \infty$ . This entails [15]  $\lim_{l \rightarrow \infty} \lim_{N \rightarrow \infty} \langle m_k, \mathbf{V}(\mathbf{A})^l \phi_0 \rangle = \lim_{N \rightarrow \infty} \langle m_k, \phi_0 \rangle = 0$ .

Moreover, by (5) the specific free energy  $f = \beta \Phi/N$  is the same for all eigenvectors  $m_k$ : the third term depends only on  $q$ , the second is  $-\beta q/2$  because  $Jm_k = m_k$  and

$$2N^{-1} \sum_{i=1}^N \{ (1 + m_i) \ln(1 + m_i)/2 + (1 - m_i) \ln(1 - m_i)/2 \} = q + O(q^{3/2})$$

since  $q$  is by definition small near the critical point. We conclude with two remarks.

(i) The staggered magnetizations can be explicitly computed for some particular values of  $N$ . There is indeed [10]  $p(N) \in \mathbb{N}$  such that  $\mathbf{J}(\mathbf{A})^{p(N)} = I_d$ , and for ‘most’ sequences of values of  $N$  one has  $p(N)/N = M, M < \infty$ . Under these conditions the eigenvalue 1 of  $\mathbf{J}(\mathbf{A})$  has multiplicity  $M + 1$ . A first corresponding (norm  $\epsilon$ ) eigenvector is [9]  $m_1 = \cos(2\pi i/N) \bar{k} l^2 / \sqrt{\epsilon N}$  where  $\bar{k}^2 = 3 \pmod{N}$ , while

$$m_l^r = 1/\sqrt{\epsilon p} \sum_{s=0}^{p-1} c_s^r \cos(2\pi i/N) (a_s^r l^2 + b_s^r l) / \sqrt{N} \quad r = 2, \dots, M + 1$$

where  $|c_s^r| \leq 1$  and  $0 \leq a_s^r, b_s^r \leq N - 1$  are integers ([9, formula (4.25)]). By (9) one checks directly that  $N^{-1} \sum_{l=0}^{N-1} m_l^i \rightarrow 0$  for all eigenvectors.

(ii) The magnetization just below the critical point has the square-root behaviour of a second-order transition. Compute indeed the second-order expansion  $\delta^2 Q_i$  near  $\beta_c$  and  $m^*$  as in [5], putting  $\beta = \beta_c + \Delta\beta$  and  $m_i = m_i^* + \delta m_i$  where  $m^*$  is any staggered magnetization vector and  $\langle \delta m, m^* \rangle = 0$ . Neglecting the term of order  $\Delta\beta^2$ , taking the scalar product with  $m^*$  and dividing by  $Nq$  the equation  $\langle m^*, \delta^2 Q \rangle = 0$  we obtain  $\alpha \|\delta m\|^2 + \gamma \Delta\beta = 0$  with  $\alpha = -2\beta_c^3 G''(\beta_c) > 0, \gamma = -1 + 2G'(\beta_c) + 2\beta_c G''(\beta_c) < 0$ . Hence the assertion.

(iii) As those of [1–3], the present model is long range and its aim is to reproduce at least some of the main features of the random models solved with the spontaneously broken replica symmetry formalism [12]. The agreement with more realistic spin-glass models is discussed in [16, 17].

We thank G Parisi for a useful conversation.

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